Name:………………………………………………………………………………….

Student Number: ………………………………………………………………………..

Re-Exam Electricity and Magnetism

Content: 12 pages (including this cover page)

Wednesday 11 July 2018; A. Jacobshal 01, 9:00-12:00

- Write your full name and student number
- Write your answers in the designated area; if you use **extra sheets, indicate this clearly!**
- Reversed sides of each page are left blank intentionally and could be used for draft answers
- Read the questions carefully
- Please, do not use a pencil
- Books, notes, phones, smartphones and tablets are not allowed. Calculators and dictionaries are allowed.

The weighting of the questions and the grading scheme (total number of points, and mention the number of points at each (sub)question).
Grade = 1 + 9 x (score/max score).

For administrative purposes; do NOT fill the table

	Maximum points	Points scored
Question 1	15	
Question 2	20	
Question 3	10	
Question 4	10	
Question 5	15	
Total	**70**	

Final mark: __________

Question 1. (15 points)

A. Find the capacitance per unit length of two coaxial metal cylindrical tubes, of radii a and b. (5 points)

B. A long straight wire, carrying a uniform line charge λ, is surrounded by rubber insulation out to a radius a. Find the electric displacement (it's a vector!). (5 points)

C. Find the total force (including direction) per unit length between two long parallel wires (a distance d apart) with the current flowing in opposite directions. (5 points)

Answers:

Model answers Question 1: (15 points)

A. Problem 2.43 (p107) (5 points)

Problem 2.43

Say the charge on the inner cylinder is Q, for a length L. The field is given by Gauss's law: $\int \mathbf{E}\cdot d\mathbf{a} = E \cdot 2\pi s \cdot L = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}Q \Rightarrow \mathbf{E} = \frac{Q}{2\pi\epsilon_0 L}\frac{1}{s}\hat{\mathbf{s}}$. Potential difference between the cylinders is

$$V(b) - V(a) = -\int_a^b \mathbf{E}\cdot d\mathbf{l} = -\frac{Q}{2\pi\epsilon_0 L}\int_a^b \frac{1}{s}ds = -\frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right).$$

As set up here, a is at the higher potential, so $V = V(a) - V(b) = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)$.

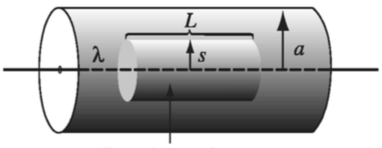
$C = \frac{Q}{V} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)}$, so capacitance *per unit length* is $\boxed{\frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}}$.

B. Example 4.4 (p182) (5 points)

Solution

Drawing a cylindrical Gaussian surface, of radius s and length L, and applying Eq. 4.23, we find

$$D(2\pi s L) = \lambda L.$$



Therefore,

$$\mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}. \tag{4.24}$$

Notice that this formula holds both within the insulation and outside it. In the latter region, $\mathbf{P} = 0$, so

$$\mathbf{E} = \frac{1}{\epsilon_0}\mathbf{D} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}, \quad \text{for } s > a.$$

Inside the rubber, the electric field cannot be determined, since we do not know **P**.

C. Chapter 5, (p226) (5 points)

As an application, let's find the force of attraction between two long, parallel wires a distance d apart, carrying currents I_1 and I_2 (Fig. 5.20). The field at (2) due to (1) is

$$B = \frac{\mu_0 I_1}{2\pi d},$$

and it points into the page. The Lorentz force law (in the form appropriate to line currents, Eq. 5.17) predicts a force directed towards (1), of magnitude

$$F = I_2\left(\frac{\mu_0 I_1}{2\pi d}\right)\int dl.$$

The *total* force, not surprisingly, is infinite, but the force per unit length is

$$f = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}. \tag{5.40}$$

If the currents are antiparallel (one up, one down), the force is repulsive—consistent again with the qualitative observations in Sect. 5.1.1.

Question 2. (20 points)

A. How much work does it take to assemble four negative charges on the corners of a square (side a)? (5 points)

B. An infinite solenoid (n turns per unit length, current I) is filled with linear diamagnetic material of susceptibility χ_m. Find the magnetic field inside the solenoid. Is the field enhanced or reduced by the diamagnetic material? (5 points)

C. Find the exact magnetic field a distance z above the center of a square loop of side w, carrying a current I. Verify that it reduces to the field of a dipole, with the appropriate dipole moment, when $z \gg w$. Use the coordinates as depicted on the right in Figure 1. (10 points)

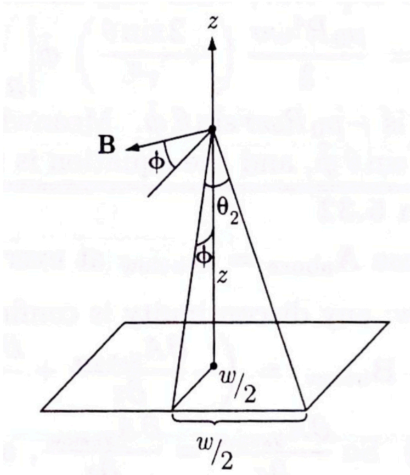


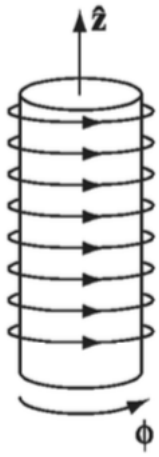
Figure 1.

Answers:

Model answers Question 2: (20 points)

A. Modified Problem 2.31 (p92-p93) (5 points)
If the reference is set at infinity, $W = QV(\boldsymbol{r})$. To place the first charge doesn't cost any energy, but the second charge costs $W_2 = \frac{1}{4\pi\epsilon_0} q \frac{q}{a}$, the third charge adds $W_3 = \frac{1}{4\pi\epsilon_0} q \left(\frac{q}{a} + \frac{q}{\sqrt{2}a}\right)$ and the last charge adds $W_4 = \frac{1}{4\pi\epsilon_0} q \left(\frac{2q}{a} + \frac{q}{\sqrt{2}a}\right)$.

B. Example 6.3 (p286) (5 points)



Since **B** is due in part to bound currents (which we don't yet know), we cannot compute it directly. However, this is one of those symmetrical cases in which we can get **H** from the free current alone, using Ampère's law in the form of Eq. 6.20:

$$\mathbf{H} = nI\,\hat{\mathbf{z}}$$

(Fig. 6.22). According to Eq. 6.31, then,

$$\mathbf{B} = \mu_0(1 + \chi_m)nI\,\hat{\mathbf{z}}.$$

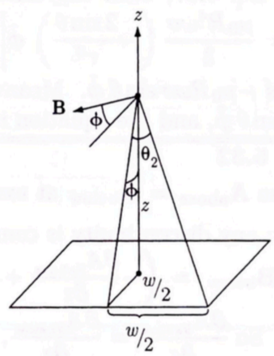
If the medium is paramagnetic, the field is slightly enhanced; if it's diamagnetic, the field is somewhat reduced. This reflects the fact that the bound surface current

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = \chi_m(\mathbf{H} \times \hat{\mathbf{n}}) = \chi_m nI\,\hat{\boldsymbol{\phi}}$$

is in the same direction as I, in the former case ($\chi_m > 0$), and opposite in the latter ($\chi_m < 0$).

C. Problem 5.36 (p255) (10 points)

Problem 5.36



The field of one side is given by Eq. 5.37, with $s \to \sqrt{z^2 + (w/2)^2}$ and $\sin\theta_2 = -\sin\theta_1 = \frac{(w/2)}{\sqrt{z^2 + w^2/2}}$;

$B = \frac{\mu_0 I}{4\pi} \frac{w}{\sqrt{z^2 + (w^2/4)}\sqrt{z^2 + (w^2/2)}}$. To pick off the vertical component, multiply by $\sin\phi = \frac{(w/2)}{\sqrt{z^2 + (w/2)^2}}$; for all four sides, multiply by 4: $\boxed{\mathbf{B} = \frac{\mu_0 I}{2\pi} \frac{w^2}{(z^2 + w^2/4)\sqrt{z^2 + w^2/2}}\,\hat{\mathbf{z}}.}$

For $z \gg w$, $\mathbf{B} \approx \frac{\mu_0 I w^2}{2\pi z^3}\,\hat{\mathbf{z}}$. The field of a dipole $\boxed{m = Iw^2,}$ for points on the z axis (Eq. 5.88, with $r \to z$, $\hat{\mathbf{r}} \to \hat{\mathbf{z}}$, $\theta = 0$) is $\mathbf{B} = \frac{\mu_0}{2\pi}\frac{m}{z^3}\,\hat{\mathbf{z}}$. ✓

Question 3. **(10 points)**
Suppose $V = 0$ and $\mathbf{A} = A_0 sin(kx - \omega t)\,\hat{\mathbf{y}}$, where A_0, ω, and k are constants.

A. Find electric **E** and magnetic **B** fields. (3 points)
B. Check that **E** and **B** satisfy Maxwell's equations in vacuum. (6 points)
C. What condition must you impose on ω and k? (1 point)

Answers:

Model answers Question 3 (Problem 10.4): (10 points)

A. 1+2 points; 3 points

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -A_0 \cos(kx - \omega t)\,\hat{\mathbf{y}}(-\omega) = \boxed{A_0\omega \cos(kx - \omega t)\,\hat{\mathbf{y}},}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \hat{\mathbf{z}}\,\frac{\partial}{\partial x}\left[A_0 \sin(kx - \omega t)\right] = \boxed{A_0 k \cos(kx - \omega t)\,\hat{\mathbf{z}}.}$$

B. 6 points

$\nabla\cdot\mathbf{E} = 0$ ✓, $\nabla\cdot\mathbf{B} = 0$ ✓ 1 point each

$$\nabla\times\mathbf{E} = \hat{\mathbf{z}}\,\frac{\partial}{\partial x}\left[A_0\omega \cos(kx - \omega t)\right] = -A_0\omega k \sin(kx - \omega t)\,\hat{\mathbf{z}}, \quad -\frac{\partial \mathbf{B}}{\partial t} = -A_0\omega k \sin(kx - \omega t)\,\hat{\mathbf{z}},$$

so $\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ ✓. 2 points

$$\nabla\times\mathbf{B} = -\hat{\mathbf{y}}\,\frac{\partial}{\partial x}\left[A_0 k \cos(kx - \omega t)\right] = A_0 k^2 \sin(kx - \omega t)\,\hat{\mathbf{y}}, \quad \frac{\partial \mathbf{E}}{\partial t} = A_0\omega^2 \sin(kx - \omega t)\,\hat{\mathbf{y}}.$$

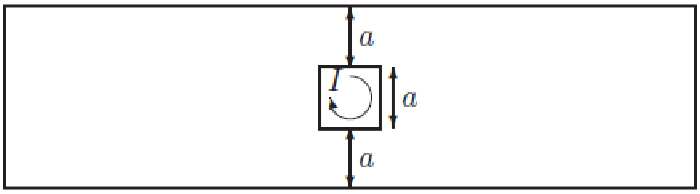
2 points

C. 1 point

$$\text{So } \nabla\times\mathbf{B} = \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} \textit{ provided } \boxed{k^2 = \mu_0\epsilon_0\omega^2,} \text{ or, since } c^2 = 1/\mu_0\epsilon_0, \boxed{\omega = ck.}$$

Question 4 (10 points)

A square loop of wire, of side a, lies midway between two long wires, $3a$ apart, and in the same plane. (Actually, the long wires are sides of a large rectangular loop, but the short ends are so far away that they can be neglected.) A clockwise current I in the small square loop is gradually increasing: $dI/dt=k$ (a constant).



A. Find the mutual inductance of the loops. Tip: you might find useful to exploit the equality of the mutual inductances. (5 points)
B. Find the emf induced in the big loop. (3 points)
C. Which way will the induced current flow? (2 points)

Answers:

Model answers Question 4 (Problem 7.23): (10 points)

A. (5 points)

It's hard to calculate M using a current in the little loop, so, exploiting the equality of the mutual inductances, I'll find the flux through the *little* loop when a current I flows in the *big* loop: $\Phi = MI$. The field of *one* long wire is $B = \frac{\mu_0 I}{2\pi s} \Rightarrow \Phi_1 = \frac{\mu_0 I}{2\pi} \int_a^{2a} \frac{1}{s} a\, ds = \frac{\mu_0 I a}{2\pi} \ln 2$, so the *total* flux is

$$\Phi = 2\Phi_1 = \frac{\mu_0 I a \ln 2}{\pi} \Rightarrow M = \frac{\mu_0 a \ln 2}{\pi}$$

B. (3 points)

$$\mathcal{E} = -\frac{d\Phi}{dt} = -M\frac{dI}{dt} = -Mk.$$

$$\boxed{\mathcal{E} = \frac{\mu_0 k a \ln 2}{\pi},}$$

C. (2 points)

Direction: The net flux (through the big loop), due to I in the little loop, is *into the page*. (Why? Field lines point *in*, for the inside of the little loop, and *out* everywhere outside the little loop. The big loop encloses *all* of the former, and only *part* of the latter, so *net* flux is *inward*.) This flux is *increasing*, so the induced current in the big loop is such that *its* field points *out* of the page: it flows $\boxed{\text{counterclockwise.}}$

Question 5 (15 points)

A plane electromagnetic wave $E_0 cos(kz - \omega t)$ travelling through vacuum in the positive z direction and polarized into the x direction, encounters a perfect conductor, occupying the region $z \geq 0$, and reflects back. The electric field inside a perfect conductor is zero.

A. Find the complete electric field of the plane electromagnetic wave in the $z < 0$ region, by invoking the proper boundary condition (see the formula sheet). (5 points)

B. Find the accompanying magnetic field in the $z < 0$ region. (5 points)

C. Assuming **B=0** inside the conductor, find the current **K** on the surface z=0 by invoking the appropriate boundary condition (2 points)

D. Find the magnetic force **f** (averaged per time) per unit area at the surface (Tip: $\mathbf{f} = \mathbf{K}\times\mathbf{B}$) (2 points)

E. Does you answer make any sense? (1 point)

Answers

<u>Model Answers Question 5 (Griffiths, Problem 9.34 modified) (15 points)</u>

A. Because the EM wave orthogonal to the interface, the boundary condition

$\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0$

$\mathbf{E}_2^{\parallel} = 0$ because the conductor is perfect (1 point)

$E_I + E_R = 0$; $E_R = -E_I$ - the reflected wave has a π phase shift (1 point)

$\mathbf{E} = E_0[cos(kz - \omega t) - cos(kz + \omega t)]\hat{\mathbf{x}}$ (2 points)

(-1 point if no $\hat{\mathbf{x}}$)

B. $\mathbf{B} = \frac{E_0}{c}[cos(kz - \omega t) + cos(kz + \omega t)]\hat{\mathbf{y}}$ (4 points)

$B_0 = \frac{E_0}{c}$ because of scaling of the magnetic field (-1 point if incorrect)

$\hat{\mathbf{y}}$ because of polarization along $\hat{\mathbf{x}}$ and propagation along $\hat{\mathbf{z}}$ (-1 point if incorrect)

The "+" sign because **E** changes the sign upon reflection so **B** does not, and **E**×**B** is directed to the propagation direction (-1 point if incorrect)

C. (2 points)

$$\mathbf{K} \times (-\hat{\mathbf{z}}) = \frac{1}{\mu_0}\mathbf{B} = \frac{E_0}{\mu_0 c}[2\cos(\omega t)]\,\hat{\mathbf{y}},\ \ \mathbf{K} = \boxed{\frac{2E_0}{\mu_0 c}\cos(\omega t)\,\hat{\mathbf{x}}.}$$

D. The force per unit area is

$$\mathbf{f} = \mathbf{K} \times \mathbf{B}_{\text{ave}} = \frac{2E_0^2}{\mu_0 c^2}[\cos(\omega t)\,\hat{\mathbf{x}}] \times [\cos(\omega t)\,\hat{\mathbf{y}}] = \boxed{2\epsilon_0 E_0^2 \cos^2(\omega t)\,\hat{\mathbf{z}}.}$$

The time average of $\cos^2(t)$ is 1/2, so (2 points)

$$\mathbf{f}_{\text{ave}} = \boxed{\epsilon_0 E_0^2.}$$

E. This is exactly radiation pressure at a perfect reflector (1 point)

Maxim S. Pchenitchnikov **Steven Hoekstra**

9 July 2018 9 July 2018